Section 13.3
Arc Length and Speed

by Joseph Phillip Brennan
Jila Niknejad

1 Arc Length and Speed

Arc Length

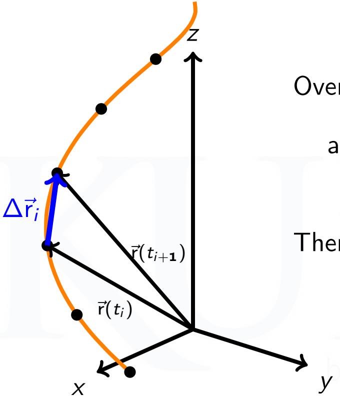
Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ be a parametrization of a smooth curve $\mathcal{C}$.

We want to measure the **arc length** of $\mathcal{C}$ for t in $[a, b]$.

Equivalently, if $\vec{r}(t) =$ position of a particle at time t, then we would like to know the **total distance traveled** by the particle during the time interval $[a, b]$.

(Note: Arc length is **not** the same as *displacement* $\|\vec{r}(b) - \vec{r}(a)\|$, unless $\mathcal{C}$ is a line.)

Example 1: $\vec{r}(t) = \langle R\cos(t),\ R\sin(t), 0 \rangle$. Here $\mathcal{C}$ is a circle of radius R and the distance traveled on $[a, b]$ is $R(b - a)$.



Partition $[a, b]$ into n subintervals of size $\Delta t = \frac{b-a}{n}$.

$$a = t_0 < \ldots < t_i < t_{i+1} < \ldots < t_n = b.$$

Over each interval $[t_i, t_{i+1}]$,

$$\text{arc length} \approx \underbrace{\|\vec{r}(t_{i+1}) - \vec{r}(t_i)\|}_{\text{The chord's length}} \approx \|\vec{r}\,'(t_i)\|\,\Delta t.$$

Therefore, the arc length is given by a Riemann sum:

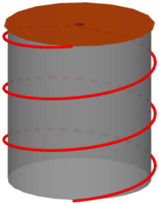
$$\lim_{n\to\infty} \sum_{i=1}^{n} \|\vec{r}\,'(t_i)\|\,\Delta t.$$

Arc Length Formula

The arc length of the curve parametrized by $\vec{r}(t)$ for $a \le t \le b$ is

$$\int_a^b \|\vec{r}\,'(t)\|\,dt = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\,dt.$$

Arc Length



Example 2: A cylindrical tower is 30 feet high and has a radius of 10 feet. A spiral staircase wraps around the tower three times from the base to the top of the tower. How long is the staircase?

Solution: The staircase is a helix, so we can parametrize it as follows:

$$\vec{r}(t) = \langle 10\cos(t),\, 10\sin(t),\, 30t/6\pi \rangle \qquad 0 \le t \le 6\pi$$

$$\vec{r}\,'(t) = \langle -10\sin(t),\, 10\cos(t),\, 5/\pi \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{(-10\sin(t))^2 + (10\cos(t))^2 + \left(\tfrac{5}{\pi}\right)^2} \quad = \frac{\sqrt{100\pi^2 + 25}}{\pi}$$

Length of the staircase: $\displaystyle\int_0^{6\pi} \|\vec{r}\,'(t)\|\, dt = 6\sqrt{100\pi^2 + 25} \approx 77.92$ feet.

2 The Arc Length Function and Arc Length Parameterization

The Arc Length Function

The length of the portion of the curve $\vec{r}$ over the interval $[a, t]$ is

$$s(t) = \int_a^t \|\vec{r}\,'(\tau)\|\, d\tau$$

The letter s is reserved for arc length. (Here τ is just a dummy variable.) Note that $s(t)$ is a scalar function of t.

- Speed = rate of change of distance traveled with respect to time t
- Speed = magnitude of velocity
- To see that these two definitions are consistent, differentiate the arc length formula with respect to time t, using FTC:

$$\text{Speed at time } t \;= \frac{ds}{dt} = \dot{s} = \|\vec{r}\,'(t)\|.$$

Arc Length Is Intrinsic

Fact: Arc length is an **intrinsic** property of a curve $\mathcal{C}$.

Even though we need a parametrization to use the arc length formula, the arc length depends **only on $\mathcal{C}$ itself**, not on the choice of parameterization.

In other words, if $\mathcal{C}$ is parametrized by two separate functions $\vec{r}(t)$ for $a \le t \le b$ and $\vec{q}(u)$ for $c \le u \le d$, then

$$\text{arc length} \;=\; \int_a^b \|\vec{r}\,'(t)\|\,dt \;=\; \int_c^d \|\vec{q}\,'(u)\|\,du.$$

Example 3:
$\vec{r}(t) = \langle \cos(mt), \sin(mt) \rangle$ for $t \in [0, 2\pi/m]$ parameterizes the unit circle for any $m \in \mathbb{R} - \{0\}$.

$$\int_0^{2\pi/m} \|\vec{r}\,'(t)\|\,dt = \int_0^{2\pi/m} \|\langle -m\sin(mt),\, m\cos(mt)\rangle\|\,dt = \int_0^{2\pi/m} m\,dt$$
$$= 2\pi \quad \underline{\text{(independently of } m)}.$$

Arc Length Parametrization

- Think of a smooth curve $\mathcal{C}$ as a road. A parametrization $\vec{r}(t)$ describes how a car drives along the road.
- It would not be very useful to measure a road by having a car drive along it randomly and place a marker every minute!
- Much more useful would be to place **mile markers** — that is, to record the value of s along the road.
- The mile markers themselves provide a parametrization, namely

$$\vec{r}(s) = \text{ point in the road } s \text{ miles from a fixed starting point.}$$

The parameter is arc length (s) rather than time, so this is called an **arc length parametrization.** It has the special property that

$$\|\vec{r}\,'(s)\| = \frac{ds}{ds} = 1,$$

so an arc length parametrization could also be defined as a parametrization with constant speed 1.

Arc Length Parametrization: Example

Example 4: Find an arc length parametrization for the helix

$$\vec{\mathrm{r}}(t) = \langle \cos(t), \sin(t), t \rangle .$$

<u>Solution:</u> Note that $\vec{\mathrm{r}}'(t) = \langle -\sin(t), \cos(t), 1 \rangle$ and $\|\vec{\mathrm{r}}'(t)\| = \sqrt{2}$.
The arc length function s, starting at $t = 0$, is

$$s(t) \;=\; \int_0^t \sqrt{2}\, d\tau \;=\; \sqrt{2}t \quad \therefore \quad t = \frac{s}{\sqrt{2}}.$$

In this case, all we have to do to find an arc length parametrization is to replace t with $s/\sqrt{2}$:

$$\vec{\mathrm{q}}(s) = \vec{\mathrm{r}}\left(\frac{s}{\sqrt{2}}\right) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right),\ \sin\left(\frac{s}{\sqrt{2}}\right),\ \frac{s}{\sqrt{2}} \right\rangle .$$

However, it is often impossible to find t explicitly as a function of s.

Arc Length Parametrization: Example

Example 5: Find an arc length parametrization for the curve

$$\vec{r}(t) = \left\langle t,\ t^2,\ t^3 \right\rangle .$$

Solution: Note that $\vec{r}\,'(t) = \left\langle 1,\ 2t\ , 3t^2 \right\rangle$ and $\|\vec{r}\,'(t)\| = \sqrt{1+4t^2+9t^4}$. The arc length function s, starting at $t=0$, is

$$s(t) \ = \ \int_0^t \sqrt{1+4\tau^2+9\tau^4}\ d\tau.$$

This is a well-defined, increasing (hence invertible!) function of t, but neither it nor its inverse can be expressed in any reasonable closed form.

Nevertheless, the arc length parametrization **exists**, and we can often work with it (using FTC and the Chain Rule) without needing to write it down as an explicit formula.

Arc Length Parametrization in General

Any parametrization $\vec{r}(t)$ of a smooth curve $\mathcal{C}$ can be used to find an arc length parametrization of the curve:

Since $s'(t) = \|\vec{r}\,'(t)\| > 0$, the function $s(t)$ is increasing, therefore has an inverse. So we can think of t as a function of s.

With this in mind, define a parametrization $\vec{q}(s)$ as a composition:

$$\vec{q}\,(s) = \vec{r}\big(t(s)\big)$$

The function $\vec{q}$ has the same image as $\vec{r}$, so it also parametrizes $\mathcal{C}$. Moreover, $\vec{q}$ is an arc length parametrization because

$$\left\|\frac{d\vec{q}}{ds}\right\| = \left\|\frac{d\vec{r}}{dt}\right\| \frac{dt}{ds} = \|\vec{r}\,'(t)\| \, \frac{1}{\|\vec{r}\,'(t)\|} = 1$$

In particular, $\vec{q}\,'(s)$ is a **unit tangent vector** for all s.

What We're Skipping in Chapter 13

Some of the things you can do with an arc length parametrization:

- construct the moving frame (which measures how the directions "forward", "inward" and "upward" change along the curve)
- measure curvature (how bendy is a curve, i.e., how much does it deviate from linearity?)
- measure torsion (how far much does it deviate from lying in a plane?)
- with a little basic physics, derive Kepler's laws of planetary motion (the first largely accurate model for the astronomical universe, and it's all based on vectors and calculus)